Classification of Measurements. Several measurements on the same or different quantities are independent when (1) no mathematical relation necessarily exists among them and (2) the different measurements are entirely unbiased by each other or by other results.

Measurements that satisfy the second condition, but not the first, are known as "conditioned measurements." Thus, in the complete chemical analysis of a material, a mathematical relation must exist among the percentages of the various constituents, inasmuch as these must add up to 100.

Measurements satisfying the first, but not the second condition, are said to be "dependent." It is highly desirable, but often difficult, to avoid dependent measurements. For instance, when a series of readings are being made by resetting the indicator on a scale, if the previous results are remembered, one must contend with the temptation to make the new readings agree closely with the previous results rather than to exercise completely independent judgment. Such a practice renders it impossible to determine the true precision of the observations, since the various readings are not truly independent but are affected by preconceived notions.

All measurements may be classed as either direct or indirect. A direct measurement is made whenever the magnitude of the measured quantity is determined by direct observation from the measuring instrument. The measurement of length by a meter stick, time by a clock, and weight by a balance are examples of direct measurements.

In contrast to this direct procedure, the magnitude of a quantity is often measured by calculation from the magnitudes of other quantities directly measured, the calculation being made by means of some functional relationship existing among the quantities. The estimate of error in an indirect measurement is more difficult than the estimate of error in a direct measurement, since the errors in the direct measurements concerned may either augment or offset each other's effect on the error of the calculated result, depending upon their signs and the form of the functional relationship. Indirect measurements may be made for the purpose of computing a desired quantity from a group of directly measurable quantities by means of a known functional relationship containing known constants or for the purpose of determining the unknown constants in a functional relationship of known form.

Propagation of Errors. When the desired quantity M is related to the several directly measured quantities $M_1, M_2, \ldots, M_n$ by the equation

$$M = \gamma(M_1, M_2, \ldots, M_n) \tag{2-2}$$

M becomes an indirectly measured quantity. In general, the true value of M cannot be known because the true values of $M_1, M_2, \ldots, M_n$ are unknown, but the most probable value of M, denoted by Q, may be calculated by inserting the most probable values of $M_1, M_2, \ldots, M_n$, denoted by $q_1, q_2, \ldots, q_n$, into (2-2). Evidently, the errors in the directly measured quantities will result in an error in the calculated quantity, the value of which it is important to ascertain. If the original measurements are available, an obvious method of procedure would be to calculate a value of M corresponding to every set of measurements. The mean of all these calculated values could then be obtained and the characteristic errors of the mean calculated from the residuals.

Very often, however, the only data available are $q_1, q_2, \ldots, q_n$, together with their characteristic errors, from which it is necessary to estimate the characteristic errors in Q. It may be that $q_1, q_2, \ldots, q_n$ are the most probable values calculated from a set of observations, or they may be merely estimated values employed in the preliminary discussion of the proposed measurement. In these cases, it becomes necessary to devise a procedure for relating the errors in the measured quantities to the error in the calculated quantity.

Such a procedure makes possible the solution of the two fundamental problems of indirect measurements:

1. Given the errors of several directly measured quantities, to calculate the error of any function of these quantities
2. Given a prescribed error in the quantity to be indirectly measured, to specify the allowable errors in the directly measured quantities

The method is as follows: In terms of the most probable quantities, (2-2) may be written

$$Q = \gamma(q_1, q_2, \ldots, q_n) \tag{2-3}$$

The differential change in Q corresponding to a differential change in each of the q's is

$$dQ = \frac{\partial \gamma}{\partial q_1} dq_1 + \frac{\partial \gamma}{\partial q_2} dq_2 \cdots \frac{\partial \gamma}{\partial q_n} dq_n \tag{2-4}$$

where $\partial\gamma/\partial q_n$ denotes the *partial* derivative of γ with respect to q_n and is obtained by differentiating γ with respect to q_n, with all other q's regarded as constant.

If the differentials $dq_1, dq_2, \ldots, dq_n$ are replaced by small finite increments $\Delta q_1, \Delta q_2, \ldots, \Delta q_n$, there results as a good approximation†

† The limitations on this approximation by means of the first differential may become clearer from the following considerations: Errors of $\Delta q_1, \Delta q_2, \ldots, \Delta q_n$ in the quantities $q_1, q_2, \ldots, q_n$ will produce a corresponding error ΔQ in Q according to the equation

$$Q + \Delta Q = \gamma(q_1 + \Delta q_1, q_2 + \Delta q_2, \ldots, q_n + \Delta q_n)$$

Expansion of γ in the neighborhood of $q_1, q_2, \ldots, q_n$ by means of Taylor's theorem gives

$$Q + \Delta Q = \gamma(q_1, q_2, \ldots, q_n) + \frac{\partial \gamma}{\partial q_1}\Delta q_1 + \frac{\partial \gamma}{\partial q_2}\Delta q_2 + \cdots + \frac{\partial \gamma}{\partial q_n}\Delta q_n + \frac{\partial^2 \gamma}{\partial q_1^2}\frac{(\Delta q_1)^2}{1 \cdot 2} + \cdots$$

(Terms of higher order)

If the quantities Δq are small, the terms of higher order are negligible, and the expression reduces to (2-5).

for ΔQ the expression

$$\Delta Q = \frac{\partial \gamma}{\partial q_1}\,\Delta q_1 + \frac{\partial \gamma}{\partial q_2}\,\Delta q_2 + \cdots + \frac{\partial \gamma}{\partial q_n}\,\Delta q_n \qquad (2\text{-}5)$$

The quantities $\Delta q_1, \Delta q_2, \ldots, \Delta q_n$ may be considered as errors in $q_1, q_2, \ldots, q_n$, and (2-5) provides a means of computing the resulting error in the function. Equation (2-5) holds for any type of errors, provided only that they are small. On the other hand, (2-5) does not utilize all the information that may be available and consequently often *overestimates* the error in Q. The following example will illustrate the use of (2-5) and also point out its defects.

Example During the course of an analysis of plant performance it becomes necessary to determine the average velocity of water flowing through a certain pipe. The most convenient method of measurement is an indirect one: measurement of the weight W of water issuing from the pipe during the time t, measurement of the pipe diameter D, and calculation of the average velocity from the density ρ of the water and the relation

$$V_{av} = \frac{W}{tA\rho} \cong \frac{4W}{\pi D^2 t\rho} \qquad (2\text{-}6)$$

Before undertaking the measurement, the engineer decides to calculate the uncertainty in his result by means of the procedure based upon Eq. (2-5). Accordingly, he estimates the values of the variables and their uncertainty as follows:

1. Weight of water. Information concerning the weighing scales available sets 100 lb as a convenient figure for the weight of water to be collected. The particular scale to be used is not yet known. However, the engineer recognizes that some of the scales in the plant are in rather poor repair and, on the chance that one of these might be used, takes a "conservative" uncertainty of ± 5 lb.
2. Time of collection of sample. Previous information indicates that approximately 70 sec will be required to collect the 100-lb sample. An electric clock will be used to measure the elapsed time. The engineer feels that human and clock error combined will not exceed ± 1 sec.
3. The pipe area. The nominal diameter of the pipe is 1 in. Taking into account the deviations from roundness, caliper error, etc., the engineer mentally estimates an uncertainty in the pipe diameter not exceeding ± 0.03 in.
4. Water density. The water temperature will be about 60°F. The density at 60°F is 62.34 pcf. The estimated uncertainty in the temperature is ± 3°F corresponding to a density variation of less than 0.1 per cent. Since the uncertainty in the water density is an order of magnitude less than that of the other quantities, its effect may be neglected.

Approximate values of the data and the estimated maximum errors are then the following:

Variable	Approximate value	Measured to
W	100 lb	± 5 lb
t	70 sec	± 1.0 sec
D	1 in.	± 0.03 in.

The partial derivatives of (2-6) needed for use in an equation of the form of (2-5) are

$$\frac{\partial V_{av}}{\partial W} = \frac{4}{t\pi D^2\rho} \qquad \frac{\partial V_{av}}{\partial t} = -\frac{4W}{\pi D^2 \rho t^2} \qquad \frac{\partial V_{av}}{\partial D} = -\frac{8W}{t\pi\rho D^3}$$

Consequently,

$$\begin{aligned}\Delta V &= \frac{4}{t\pi D^2\rho}\Delta W - \frac{4W}{\pi D^2\rho t^2}\Delta t - \frac{8W}{t\pi\rho D^3}\Delta D \\ &= \frac{4 \cdot 144}{70 \cdot 3.14 \cdot 1 \cdot 62.3}\Delta W - \frac{4 \cdot 100 \cdot 144}{3.14 \cdot 1 \cdot 62.3(70)^2}\Delta t - \frac{8 \cdot 100 \cdot 1{,}728}{70 \cdot 3.14 \cdot 62.3 \cdot 1}\Delta D \\ &= 0.042\Delta W - 0.060\Delta t - 100\Delta D \qquad (2\text{-}7)\end{aligned}$$

In order to obtain the maximum error, the sign of the ΔW will be taken as positive, and the signs of Δt and ΔD will be taken negative. Therefore,

$$\begin{aligned}\Delta V_{max} &= 0.042 \cdot 5 + 0.060 \cdot 1.0 + 100\frac{0.03}{12} = 0.210 + 0.060 + 0.252 \\ &= 0.522 \text{ fps}\end{aligned}$$

Since the approximate value of the velocity is

$$V = \frac{4 \cdot 100 \cdot 144}{70 \cdot 3.14 \cdot 1 \cdot 62.3} = 4.21 \text{ fps}$$

the maximum percentage error is $\pm(0.522/4.21)100 = \pm 12.4$ per cent.

Some important points are illustrated by the above example. It will be seen that the error in a calculated quantity that is a function of several directly measured quantities depends on (1) the nature of the function, (2) the magnitudes of the measured quantities, and (3) the magnitudes of the errors. Furthermore, variables such as ρ, whose values are known much more accurately than the rest of the variables, may be considered constants, since the error they introduce into the final result will be negligible.

It is evident that Eq. (2-5) quite probably overestimates the error involved in the measurement. It takes no account of the possibility of compensating errors. Even more serious is its failure to take into account the method used to obtain the original estimates of the uncertainties in the directly measured quantities. Presumably the engineer felt that errors exceeding those estimated were most improbable. Clearly, then, the *simultaneous* occurrence of three error extremes is distinctly less probable than the occurrence of more modest errors. For example, suppose that the probability of the assumed errors in W, t, and D in Example 2-1 are each 0.1, i.e., that 90 per cent of the time the actual errors are smaller than this. Then the probability of obtaining an error in V as large as the calculated 12.4 per cent is only one in a thousand. Modern statistical methods often permit the analyst to take such probability factors into account in the error analysis. When this is possible, the results are more realistic and valuable. Succeeding sections will discuss some of these methods.

Despite the fact that Eq. (2-5) usually overestimates the uncertainty in a dependent quantity, it is a valuable tool. In the case of formulas of the type of (2-6), consisting of the products of powers of the variables, it is possible to effect a simplification in the calculations by the use of fractional errors. To illustrate this, consider the general function

$$Q = q_a{}^a q_b{}^b \cdots q_n{}^n \tag{2-8}$$

Applying (2-5) gives

$$\Delta Q = (q_b{}^b \cdots q_n{}^n) a q_a{}^{a-1}\,\Delta q_a + (q_a{}^a q_c{}^c \cdots q_n{}^n) b q^{b-1}\,\Delta q_b \cdots$$

and

$$\frac{\Delta Q}{Q} = a\,\frac{\Delta q_a}{q_a} + b\,\frac{\Delta q_b}{q_b} + \cdots + n\,\frac{\Delta q_n}{q_n} \tag{2-9}$$

which states that the fractional error in the function Q is given by the sum of the fractional errors in the measured quantities, each multiplied by the respective power to which it appears in the function. When (2-9) is multiplied by 100, it is seen that the same rule applies to percentage errors. This rule provides a rapid solution of the preceding example.

Example It is suggested that an attempt be made to measure V_{av} in (2-6) to within an error of ± 2.0 per cent. Under this condition, what are the allowable errors in the directly measured quantities W, t, and D?

Either (2-5) or (2-9) may be used, but it is apparent that there is no unique answer to the problem as stated. More conditions are necessary. For example, the value of two errors might be fixed, whereupon the value of the third is fixed. In this general type of problem, it must be recognized that the labor and expense involved in measuring the various quantities to a given degree of accuracy are different. Ideally, those quantities which are easiest to measure should be measured the most accurately, more tolerance being allowed in the more difficult measurements, so that the required accuracy in the final result will be obtained with a minimum of expense for labor and apparatus. Because there is no general relationship between the difficulty and the accuracy of measurements, this condition cannot be given exact mathematical expression. As a starting point, it is customary to impose the condition that the errors in each of the directly measured quantities contribute equally to the error in the function. This condition is known as the "principle of equal effects."

Applied to (2-5) with the understanding that the sign of each Δq will be taken such as to make all terms of the same sign and will result in the maximum allowable error in ΔQ, the principle of equal effects gives

$$\Delta Q = n\,\frac{\partial \gamma}{\partial q_1}\,\Delta q_1 = n\,\frac{\partial \gamma}{\partial q_2}\,\Delta q_2 = \cdots = n\,\frac{\partial \gamma}{\partial q_n}\,\Delta q_n \tag{2-10}$$

In the case of a function such as (2-8), it is convenient to work with fractional errors, and the principle of equal effects reduces (2-9) to

$$\frac{\Delta Q}{Q} = na\,\frac{\Delta q_a}{q_a} = nb\,\frac{\Delta q_b}{q_b} \qquad \text{etc.} \tag{2-11}$$

Employing (2-11) in the solution of Example 2-2, there results in the case of W

$$100\,\frac{\Delta V}{V} = 2.0 = 3 \cdot 1\,\frac{\Delta W}{W}\,100$$

from which $100(\Delta W/W) = 0.7$ or ± 0.7 lb/100 lb. Similar calculations for the allowable errors in t and D give ± 0.5 sec and ± 0.003 in., respectively. These results should not be considered inflexible but should serve as a basis for deciding the optimum errors to tolerate in each quantity to ensure an error of no more than 2 per cent in the calculated velocity, with minimum labor and apparatus.

Discussions of the kind illustrated by the two preceding examples often serve to reveal that some quantity is being measured with a higher degree of precision than necessary, in view of the magnitude of the errors inherent in the other quantities concerned, or that particular attention must be focused upon the accurate measurement of a certain quantity, because of its unusually large influence on the final calculated result. Unfortunately, in many important cases the functions are quite complex, and calculations involving data known only in the form of tables and curves are necessary. The estimation of errors in such cases is more difficult and not infrequently can be accomplished only by actual repetition of the calculations and comparison of the results obtained from different sets of values of the measured quantities.

Many important design calculations necessitate graphical integrations. For example, one may face the problem of designing a liquid-liquid heat exchanger wherein both individual film coefficients vary considerably with temperature. The heat-transfer area is calculated by graphical evaluation of the integral

$$A = \int_{\Delta_1}^{\Delta_2} \frac{dq}{U\Delta} \tag{2-12}$$

where q and Δ are related by a heat balance and U, the over-all coefficient at any point, is a complex function of the liquid's physical properties, which in turn depend upon the temperature. In this case, a reliable estimate of the error in A due to a given error in the terminal temperatures is best obtained by repetition of the entire calculation, if different values for the terminal temperatures are used. It is especially important to note that the smaller the temperature difference Δ, the more serious an error of given magnitude in the measured temperatures becomes.

Particularly serious in many calculations on mass-transfer processes are errors in equilibrium data. The common calculation on a McCabe-

Thiele diagram of the number of theoretical plates required to effect a given separation with a given reflux ratio provides an instructive example of the importance of accurate equilibrium data. Figure represents a diagram for a binary mixture of low relative volatility. A small percentage error in the equilibrium data will cause a large percentage error in the vertical distance between operating line and equilibrium curve, with a resultant large percentage error in the number of

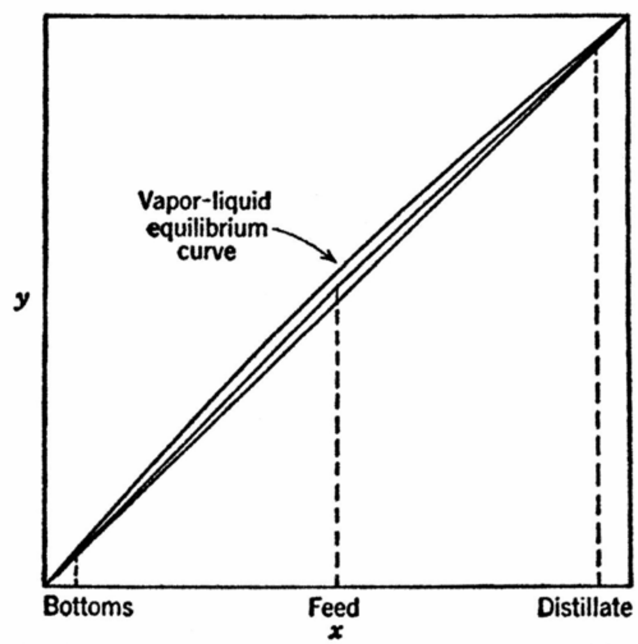

FIG. McCabe-Thiele diagram for binary system separable with difficulty.

theoretical plates. The change in the number of theoretical plates due to error in the equilibrium data is best computed by drawing in the equilibrium curve in the position corresponding to the estimated error and repeating the construction for determination of the number of theoretical plates. A similar procedure should be employed in the estimation of the error produced by equilibrium data in the calculation of the number of transfer units in adsorption, extraction, or distillation.